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Restoring boundary conditions in heat conduction

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Abstract The restoration of boundary conditions in one-dimensional transient inverse heat-conduction problems (IHCP) is described. In the formulation, the boundary conditions are represented by linear relations between the temperature and the heat flux, together with an initial condition as a function of space. The temperature inside the solution domain, together with the space or time-dependent ambient temperature of the environment surrounding the heat conductor, are found from additional boundary-temperature or average boundary-temperature measurements. Numerical results obtained using the boundary-element method are presented and discussed.

Keywords Ambient temperature · Boundary-element method · Heat conduction · Inverse problems

1 Introduction

Inverse methods have been instrumental in solving many important transient heat-transfer problems. For example, inverse problems have been formulated to resolve unspecified boundary conditions in heat conduction, [1–3], unknown initial temperature, [4,5], unknown thermophysical properties, [6–9], unknown heat sources, [10–12] and unknown heat-transfer coefficients, [13–15]. The determination of the spacewise or time-dependent ambient temperature has been theoretically investigated in [16,17], and in this paper, we investigate, for the first time, its numerical reconstruction using the boundary-element method (BEM). The application of the BEM for solving inverse heat-conduction problems has been comprehensly described in [18] for the steady state and [19] for the unsteady state (transient). Other applications of BEM inverse analyses are described in [20]. There are several advantages of using the BEM over the finite-element (FEM) or the finite-difference (FDM) methods. First, the BEM only requires a boundary mesh to discretise the problem and, as such, it is very flexible and applicable to complex geometries without having to resort to intricate internal mesh generation of unnecesary internal information as required by the traditional FDM or FEM. Second, the unknown ambient temperature, boundary temperature and heat flux are boundary quantites to be determined and the discretisation of the boundary only is the essence of the BEM. Further, the heat flux is computed as part of the solution and is not a post-processing numerical differentation. If convection occurs only on a part of the boundary of the heat conductor which may be inaccessible to measurements, then, in principle, the ambient temperature could be determined by solving a Cauchy ill-posed

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Table 1 Nomenclature

A, z_1, z_2 = quantities denoted	S_1, S_2, S_{11}, S_{22} = boundaries
b_0, b_1, h_0, h_1 = given boundary functions	t = time coordinate
B_0, B_1, F = functions	t_f = arbitrary fixed time of interest
$C^{2,1}$ = space of functions twice continuously differentiable in the space variable and once continuously differentiable in the time variable	t_j = boundary-element endpoint
	$\tilde{t}_j$ = boundary-element node
	t^0 = instant at which measurements are made
	T = temperature
C^α = space of Hölder continuous functions with exponent α	T_{0j}, T_{1j} = discretised boundary temperature
	T'_{0j}, T'_{1j} = discretised heat flux
e_0, e_1 = average boundary-temperature measurements	T^0_k = discretised initial temperature
f_0, f_1, f = ambient temperatures	x = space coordinate
g = initial temperature	x_k = cell endpoint
G = fundamental solution	$\tilde{x}_k$ = cell node
H = Heaviside function	$\tilde{\tilde{X}}, \tilde{X}, X, C, D, E$ = matrices
K = kernel	$\tilde{\tilde{Z}}, \tilde{Z}, Z, \tilde{\tilde{Y}}, \tilde{Y}, Y$ = vectors
n = outward normal to the boundary	α_n, β_m = roots of transcedental equations
N = number of boundary elements on each boundary $x = 0$ and $x = 1$	$\chi, \bar{\chi}, \chi_0, \chi_1$ = temperature measurements
	γ_0, γ_1 = constants
N_0 = number of cells	η = coefficient function
$\underline{p}, \underline{p}'$ = points in the domain	κ = regularization parameter
q = heat flux	ρ = percentage of noise
Q = solution domain	σ_0, σ_1 = heat-transfer coefficients

inverse heat-conduction problem using the measurements of the temperature and the heat flux on the remaining part of the boundary.

However, in many physical problems the measurements of the temperature and the heat flux can experience practical difficulties. Physical examples include the measurement of temperature and heat flux at a highly heated hostile boundary, the difficulty in determination of temperature over the surface of a space vehicle during the short reentry time, etc. [1]. Therefore, in order to prevent this experimental difficulty of measuring both the temperature and heat flux at the same part of the boundary, in the mathematical formulation presented in Sect. 2 we allow for the convection boundary conditions be prescribed over the whole boundary. Further, in our study, the ambient temperature is allowed to vary with space or time. Hence, a more realistic model can be proposed for the heat transfer in building enclosures, e.g., glazed surfaces, where the ambient temperature can vary from surface to surface in the building, as well as with time, depending on the local air flow patterns, e.g., type of flow, operational states of equipment, external weather conditions, etc. [21].

A nomenclature with the list of symbols used in the paper is provided in Table 1.

2 Mathematical formulation

The inverse heat-conduction problem (IHCP) under investigation is given by

$$\frac{\partial T}{\partial t}(x,t) = \frac{\partial^2 T}{\partial x^2}(x,t), \quad \text{for } (x,t) \in (0,1) \times (0,t_f] =: Q \tag{2.1}$$

$$T(x,0) = g(x), \quad \text{for } x \in [0,1], \tag{2.2}$$

$$\frac{\partial T}{\partial n}(0,t) + \sigma_0(t)T(0,t) = h_0(t)f(0,t) + b_0(t), \quad \text{for } t \in (0,t_f], \tag{2.3}$$

$$\frac{\partial T}{\partial n}(1,t) + \sigma_1(t)T(1,t) = h_1(t)f(1,t) + b_1(t), \quad \text{for } t \in (0,t_f], \tag{2.4}$$

where $t_f > 0$ is an arbitrary fixed time of interest, g is a specified function of space representing the initial temperature, σ_0, σ_1, h_0, h_1, b_0 and b_1 are specified functions of time, n is the outward normal to the boundary $\{0, 1\}$ of the heat conductor $(0, 1)$, i.e., $n(0) = -1$, $n(1) = 1$, but the function f is unknown. For simplicity we assume that there is no heat generation or loss in the system. Therefore, we study the inverse problem of restoring the unknown function f in the boundary conditions (2.3) and (2.4) of the third kind (at the boundary of the heat conductor there is a convective heat transfer (exchange) with the environment). Along with the temperature T in the domain, we seek the temperature f of the environment. A related inverse problem in which the coefficients of heat transfer σ_0 and σ_1 are unknown will be investigated in a separate work. In this paper, we investigate two situations, namely:

(i) when the function $f(x,t)$, $x \in \{0, 1\}$, $t \in (0, t_f]$ depends on x only, in which case we have to determine the constants f_0 and f_1 entering the boundary conditions (2.3) and (2.4), respectively, and
(ii) when the function $f(x,t)$, $x \in \{0, 1\}$, $t \in (0, t_f]$ depends on t only, in which case we have to determine the same function $f(t)$, entering the boundary conditions (2.3) and (2.4).

In both cases, additional information called "effect" is necessary to be measured in order to compensate for the unknown "causes" of the inverse problems. In what follows we shall distinguish between the two situations (i) and (ii) defined as Problem I and Problem II, respectively.

3 Problem I

In Problem I, the function $f(x,t)$, $x \in \{0, 1\}$, $t \in (0, t_f]$ depends on x only, i.e., $f(0,t) = f_0 =$ constant and $f(1,t) = f_1 =$ constant. We further assume that $\sigma_0(t) = \sigma_0 =$ constant and $\sigma_1(t) = \sigma_1 =$ constant. Then the boundary conditions (2.3) and (2.4) become

$$\frac{\partial T}{\partial n}(0,t) + \sigma_0 T(0,t) = f_0 h_0(t) + b_0(t) =: B_0(t), \quad \text{for } t \in (0, t_f], \tag{3.1}$$

$$\frac{\partial T}{\partial n}(1,t) + \sigma_1 T(1,t) = f_1 h_1(t) + b_1(t) =: B_1(t), \quad \text{for } t \in (0, t_f], \tag{3.2}$$

respectively. Since in the situation (i) there are two extra constants f_0 and f_1 as unknowns, we assume that two measurements of the boundary temperature at the same fixed time $t^0 \in (0, t_f]$ are available, namely

$$T(0,t^0) = \chi_0, \quad T(1,t^0) = \chi_1. \tag{3.3}$$

Alternatively, instead of (3.3) we can measure the average boundary temperature as

$$e_0 = \int_0^{t_f} T(0,t)\mathrm{d}t, \quad e_1 = \int_0^{t_f} T(1,t)\mathrm{d}t. \tag{3.4}$$

Of course, in higher dimensions the parameter estimation problem will become a function estimation problem. Conditions (3.3) and (3.4) are called terminal and integral boundary observations, respectively. Then we have the following uniqueness theorem.

Theorem 3.1 ([16]) *Suppose $g \in C^1([0, 1])$, $h_i, b_i \in C([0, t_f])$, $\sigma_i \geq 0$, and the functions $h_i > 0$ are monotone nondecreasing, $i = 0, 1$, on $(0, t_f]$. Then a solution $(T \in C^{2,1}(Q), f_0, f_1)$ to the inverse problem (2.1), (2.2), (3.1)–(3.3), or (2.1), (2.2), (3.1), (3.2) and (3.4) is unique.*

Remarks

(i) For the uniqueness of the problem (2.1), (2.2), (3.1), (3.2) and (3.4), it is sufficient to require that $\int_0^{t_f} h_i(t)\mathrm{d}t > 0$, $i = 0, 1$;
(ii) the IHCP can be recast as a Fredholm integral equation of the first kind which is a classical example of an ill-posed problem, [22];
(iii) a function T satisfying (2.1), (2.2), (3.1) and (3.2) has the representation, [23, pp. 59–69],

$$T(x,t)=\sum_{m=1}^{\infty}\mathrm{e}^{-\beta_m^2 t}K(\beta_m,x)\left[\int_0^1 K(\beta_m,x^{'})\left(g(x^{'})-z_1x^{'}-z_2\right)\mathrm{d}x^{'}\right.$$
$$\left.+\int_0^t \mathrm{e}^{\beta_m^2 t^{'}}A(\beta_m,t^{'})\mathrm{d}t^{'}+z_1x+z_2\right], \tag{3.5}$$

$$z_1=\frac{\sigma_1 g^{'}(0)+\sigma_0 g^{'}(1)+\sigma_1\sigma_0\left(g(1)-g(0)\right)}{\sigma_0+\sigma_1+\sigma_0\sigma_1},\quad z_2=\frac{g^{'}(1)+\sigma_1 g(1)-(1+\sigma_1)g^{'}(0)+\sigma_0(1+\sigma_1)g(0)}{\sigma_0+\sigma_1+\sigma_0\sigma_1},$$
$$A(\beta_m,t)=K(\beta_m,0)\left[B_0(t)+z_1-\sigma_0 z_2\right]+K(\beta_m,1)\left[B_1(t)-\sigma_1 z_2-(1+\sigma_1)z_1\right]$$
$$=K(\beta_m,0)\left[h_0(t)f_0+b_0(t)+g^{'}(1)-\sigma_0 g(0)\right]+K(\beta_m,1)\left[h_1(t)f_1+b_1(t)-g^{'}(1)-\sigma_1 g(1)\right],$$

and the kernel $K(\beta_m,x)$ is given by

$$K(\beta_m,x)=\frac{\sqrt{2}\left(\beta_m\cos(\beta_m x)+\sigma_0\sin(\beta_m x)\right)}{\sqrt{\left(\beta_m^2+\sigma_0^2\right)\left(1+\frac{\sigma_1}{\beta_m^2+\sigma_1^2}\right)+\sigma_0}},$$

where β_m are the positive roots of the transcedental equation

$$\tan(\beta)=\frac{\beta\left(\sigma_0+\sigma_1\right)}{\beta^2-\sigma_0\sigma_1}.$$

The expression (3.5) involves a complicated series expansion and in higher dimensions there is little hope it can be usable. Therefore, numerical methods which are able to discretise any multidimensional problem analogous to the one above appear more useful.

4 The BEM

It is well-known that in recent years the boundary-element method (BEM) has been established to be one of the most powerful tools in solving practical problems in science and engineering. Using the BEM, the heat equation (2.1) can be recast in the integral form [24],

$$\eta(x)T(\underline{p})=\int_{S_1}\left[G(\underline{p};\underline{p}')\frac{\partial T}{\partial n}(\underline{p}')-T(\underline{p}')\frac{\partial G}{\partial n}(\underline{p};\underline{p}')\right]\mathrm{d}S_1+\int_{S_2}T(\underline{p}')G(\underline{p};\underline{p}')\mathrm{d}S_2,\quad \underline{p}'=(x,t)\in\overline{Q}, \tag{4.1}$$

where $S_1=\{0,1\}\times(0,t_f]$, $S_2=[0,1]\times\{0\}$, $\eta(x)=1$ if $x\in(0,1)$, $\eta(0)=\eta(1)=0.5$, and

$$G(x,t;\xi,\tau)=\frac{H(t-\tau)}{2\sqrt{\pi(t-\tau)}}\exp\left(-\frac{(x-\xi)^2}{4(t-\tau)}\right), \tag{4.2}$$

where H is the Heaviside function.

4.1 Numerical discretisation

In practice the integral equation (4.1) may rarely be solved analytically and thus some form of numerical approximation is necessary.

The boundary S_1 is discretised into a series of N boundary elements, namely,

$$S_{11}=\{0\}\times(0,t_f]=\cup_{j=1}^{N}\{0\}\times(t_{j-1},t_j],\quad S_{12}=\{1\}\times(0,t_f]=\cup_{j=1}^{N}\{1\}\times(t_{j-1},t_j].$$

Also the boundary S_2 is discretised into a series of N_0 cells, namely,

$$S_2=[0,1]\times\{0\}=\cup_{k=1}^{N_0}[x_{k-1},x_k]\times\{0\}.$$

Over each time boundary element $(t_{j-1}, t_j]$ the temperature T and the heat flux $\frac{\partial T}{\partial n}$ are assumed to be constant and take their values at the mid-point $\tilde{t}_j = (t_{j-1} + t_j)/2$, i.e.,

$$T(0,t) = T(0,\tilde{t}_j) = T_{0j},\ \ T(1,t) = T(1,\tilde{t}_j) = T_{1j}, \quad \text{for } t \in (t_{j-1}, t_j], \tag{4.3}$$

$$\frac{\partial T}{\partial n}(0,t) = \frac{\partial T}{\partial n}(0,\tilde{t}_j) = T'_{0j}, \quad \frac{\partial T}{\partial n}(1,t) = \frac{\partial T}{\partial n}(1,\tilde{t}_j) = T'_{1j}, \ \text{ for } t \in (t_{j-1}, t_j]. \tag{4.4}$$

Also over each space cell $[x_{k-1}, x_k)$ the temperature T is assumed constant and takes its values at the mid point $\tilde{x}_k = (x_{k-1} + x_k)/2$, i.e.,

$$T(x,0) = T(\tilde{x}\_k, 0) = T_k^0, \quad \text{for } x \in [x_{k-1}, x_k). \tag{4.5}$$

Then using the approximations (4.3–4.5), the integral equation (4.1) can be discretised as

$$\begin{aligned}\eta(x)T(x,t) = &\sum_{j=1}^{N}\left[T'_{0j}\int_{t_{j-1}}^{t_j} G(x,t,0,\tau)\mathrm{d}\tau + T'_{1j}\int_{t_{j-1}}^{t_j} G(x,t,1,\tau)\mathrm{d}\tau\right] \\ &-\sum_{j=1}^{N}\left[T_{0j}\int_{t_{j-1}}^{t_j}\frac{\partial G}{\partial n_0}(x,t;0,\tau) + T_{1j}\int_{t_{j-1}}^{t_j}\frac{\partial G}{\partial n_1}(x,t;1,\tau)\right] \\ &+\sum_{k=1}^{N_0} T_k^0 \int_{x_{k-1}}^{x_k} G(x,t;y,0)\mathrm{d}y, \quad (x,t) \in [0,1]\times(0,t_f],\end{aligned} \tag{4.6}$$

where n_0 and n_1 represent the outward normals at the boundaries $x = 0$ and $x = 1$, respectively. Equation (4.6) can be rewritten as

$$\begin{aligned}\eta(x)T(x,t) = &\sum_{j=1}^{N}\Big[T'_{0j}C_j^0(x,t) + T'_{1j}(x,t)C_j^1(x,t) - T_{0j}D_j^0(x,t) \\ &- T_{1j}D_j^1(x,t)\Big] + \sum_{k=1}^{N_0} T_k^0 E_k(x,t), \quad (x,t) \in [0,1]\times(0,t_f],\end{aligned} \tag{4.7}$$

where the coefficients are given by

$$C_j^{\xi}(x,t) = \int_{t_{j-1}}^{t_j} G(x,t;\xi,\tau)\mathrm{d}\tau = \int_{t_{j-1}}^{t_j}\frac{H(t-\tau)}{2\sqrt{\pi(t-\tau)}}\exp\left(-\frac{(x-\xi)^2}{4(t-\tau)}\right)\mathrm{d}\tau,$$

$$D_j^{\xi}(x,t) = \int_{t_{j-1}}^{t_j}\frac{\partial}{\partial \eta_\xi}G(x,t;\xi,\tau)\mathrm{d}\tau = \int_{t_{j-1}}^{t_j}\frac{H(t-\tau)}{4\sqrt{\pi(t-\tau)^3}}|x-\xi|\exp\left(-\frac{(x-\xi)^2}{4(t-\tau)}\right)\mathrm{d}\tau,$$

$$E_k(x,t) = \int_{x_{k-1}}^{x_k} G(x,t;y,0)\mathrm{d}\xi = \int_{x_{k-1}}^{x_k}\frac{1}{2\sqrt{\pi t}}\exp\left(-\frac{(x-y)^2}{4t}\right)\mathrm{d}y,$$

where $\xi \in \{0,1\}$. These integrals can be evaluated analytically and their expressions are given by:

$$C_j^{\xi}(x,j) = \begin{cases} 0, & t \le t_{j-1} \\ \left[(t-t_{j-1})/\pi\right]^{1/2}, & t_{j-1} < t \le t_j;\ r = 0 \\ r[\exp(-z^2)/z - \pi^{1/2}\mathrm{erfc}(z)]/(2\pi^{1/2}), & t_{j-1} < t \le t_j;\ r \ne 0 \\ \left[(t-t_{j-1})^{1/2} - (t-t_j)^{1/2}\right]/\pi^{1/2}, & t_j < t;\ r = 0 \\ r[\exp(-z^2)/z - \exp(-z_1^2)/z_1 + \pi^{1/2}\{\mathrm{erfc}(z) - \mathrm{erf}(z_1)\}]/(2\pi^{1/2}), & t_j < t; r \ne 0 \end{cases}$$

$$D_j^{\xi}(x,t) = \begin{cases} 0, & t \le t_{j-1} \\ 0, & t_{j-1} < t \le t_j; \quad r = 0 \\ -\mathrm{erfc}(z)/2, & t_{j-1} < t \le t_j; \quad r \ne 0 \\ [\mathrm{erf}(z) - \mathrm{erf}(z_1)]/2, & t_j < t \end{cases}$$

$$E_k(x,t) = \frac{1}{2}\left[\mathrm{erf}\left(\frac{x - x_{k-1}}{2t^{1/2}}\right) - \mathrm{erf}\left(\frac{x - x_k}{2t^{1/2}}\right)\right],$$

where the functions erf and erfc are the error functions, and

$$r = |x - \xi|, \quad z = r\frac{(t - t_{j-1})^{-1/2}}{2}, \quad z_1 = r\frac{(t - t_j)^{-1/2}}{2}.$$

If (4.7) is applied at every node on the boundary S_1 then the following set of linear algebraic equations is obtained:

$$\sum_{j=1}^{N}\left[C_{ij}^{0\xi}T'_{0j} + C_{ij}^{1\xi}T'_{1j} - D_{ij}^{0\xi}T_{0j} - D_{ij}^{1\xi}T_{1j}\right] + \sum_{k=1}^{N_0} E_{ik}(\xi)T_k^0 = 0, \quad i = \overline{1,N}, \quad \xi \in \{0,1\}, \tag{4.8}$$

where the matrices $C^{0\xi}$,$C^{1\xi}$, $D^{0\xi}$ and $D^{1\xi}$ are defined by

$$C_{ij}^{0\xi} = C_j^{\xi}(0,\tilde{t}_i), \quad C_{ij}^{1\xi} = C_j^{\xi}(1,\tilde{t}_i),$$

$$D_{ij}^{0\xi} = D_j^{\xi}(0,\tilde{t}_i) + 0.5\delta_{ij}(1-\xi), \quad D_{ij}^{1\xi} = D_j^{\xi}(1,\tilde{t}_i) + 0.5\delta_{ij}\xi,$$

$$E_{ik}^{\xi} = E_k(\xi,\tilde{t}_i), \quad \xi \in \{0,1\}.$$

On applying the boundary conditions (3.1) and (3.2) at the nodes $(0,\tilde{t}_i)$ and $(1,\tilde{t}_i)$, respectively, for $i = \overline{1,N}$, we obtain the following equations:

$$T'_{0i} = h_{0i}f_0 + b_{0i} - \sigma_0 T_{0i}, \quad T'_{1i} = h_{1i}f_1 + b_{1i} - \sigma_1 T_{1i}, \quad i = \overline{1,N}, \tag{4.9}$$

where $h_{0i} = h_0(\tilde{t}_i)$, $h_{1i} = h_1(\tilde{t}_i)$, $b_{0i} = b_0(\tilde{t}_i)$ and $b_{1i} = b_1(\tilde{t}_i)$. Also, on applying the initial condition (2.2) at the cell nodes $(\tilde{x}_k, 0)$, for $k = \overline{1,N_0}$, the values of T_k^0 are determined, namely

$$T_k^0 = T(\tilde{x}_k, 0) = g(\tilde{x}_k) = g_k, \quad k = \overline{1,N_0}. \tag{4.10}$$

Also, instead of (3.3) and (3.4) we write, by taking $t^0 = \tilde{t}_{i_0}$ with $i_0 \in \{1,\ldots,N\}$ fixed,

$$T_{0i_0} = \chi_0, \quad T_{1i_0} = \chi_1, \tag{4.11}$$

and

$$e_0 = \sum_{i=1}^{N} T_{0i}(t_i - t_{i-1}), \qquad e_1 = \sum_{i=1}^{N} T_{1i}(t_i - t_{i-1}), \tag{4.12}$$

respectively.

The IHCP given by Eqs. (2.1), (2.2), (3.1), (3.2), and (3.3) or (3.4) reduces to its discretised version given by Eqs. (4.8)–(4.10), (4.11) or (4.12). Then the resulting system of equations becomes of the form

$$X\underline{Y} = \underline{Z}, \tag{4.13}$$

where X is a known $(4N+2)\times(4N+2)$ square matrix which contains the influence matrices C^0, C^1, D^0, D^1 and E, $\underline{Y}$ is a vector of $4N+2$ unknowns, namely T_{0j}, T_{1j}, T'_{0j}, T'_{1j}, f_0 and f_1, recast as

$$\begin{array}{lll} Y_j = T_{0j} & \text{for} & j = \overline{1,N}, \\ Y_j = T_{1(j-N)} & \text{for} & j = \overline{N+1,2N}, \\ Y_j = T'_{0(j-2N)} & \text{for} & j = \overline{2N+1,3N}, \\ Y_j = T'_{1(j-3N)} & \text{for} & j = \overline{3N+1,4N}, \\ Y_{4N+1} = f_0 & \text{and} & Y_{4N+2} = f_1, \end{array}$$

and $\underline{Z}$ is a vector of $4N+2$ known elements defined by

$$
\begin{aligned}
&Z_j = -\sum_{k=1}^{N_0} E_{jk}(0) g_k && \text{for} \quad j=\overline{1,N},\\
&Z_j = -\sum_{k=1}^{N_0} E_{(j-N)k}(1) g_k && \text{for} \quad j=\overline{N+1,2N},\\
&Z_j = b_{0(j-2N)} && \text{for} \quad j=\overline{2N+1,3N},\\
&Z_j = b_{1(j-3N)} && \text{for} \quad j=\overline{3N+1,4N},\\
&Z_{4N+1} = \chi_0 && \text{and} \quad Z_{4N+2}=\chi_1 \quad \text{for (4.11)},
\end{aligned}
$$

or,

$$Z_{4N+1} = e_0 \quad \text{and} \quad Z_{4N+2} = e_1 \quad \text{for (4.12)}.$$

For simplicity, with the assistance of (4.9) we can choose to eliminate $T_{0j}^{'}$ and $T_{1j}^{'}$, so as to reduce the system of equations to have $2N+2$ unknowns in $2N+2$ equations. This reduces the system of (4.13) considerably, to the generic form

$$\tilde{X}\underline{\tilde{Y}} = \underline{\tilde{Z}} \tag{4.14}$$

containing as unknowns $\tilde{Y}_j = T_{0j}$, $j=\overline{1,N}$, $\tilde{Y}_j = T_{1(j-1)}$, $j=\overline{(N+1),2N}$, $\tilde{Y}_{2N+1}=f_0$, $\tilde{Y}_{2N+2}=f_1$. Once the system of equations (4.14) is solved, Eq. (4.9) yields by direct substitution the heat flux and the temperature $T(x,t)$ in the solution domain is obtained explicitly using the integral equation (4.7). Of course, to determine f_0 and f_1 we could have solved the problem only on the interval $[0,1]\times[0,t^0]$, but since the boundary temperature and heat flux are also required to be determined on the whole time interval $[0,t_f]$ we have solved the problem on this whole interval. Furthermore, Eq. (4.14) over the whole interval $[0,t_f]$ is required for the integral observation (4.7).

The system of linear algebraic equations (4.14) can be solved using the Gaussian elimination method. In the event that this method fails to give satisfactory results, due to the ill-conditioning of the matrix $\tilde{X}$, one of the options would be to use regularization methods, such as the Tikhonov regularization or the truncated singular-value decomposition methods [25].

5 Numerical results and discussion for Problem I

5.1 Example 1

In this section we solve the IHCP given by (2.1) in the domain $Q=(0,1)\times(0,t_f=1]$, subject to the initial condition (2.2) of the form

$$T(x,0)=g(x)=x^2, \quad x\in[0,1], \tag{5.1}$$

the boundary conditions (3.1) and (3.2) with $\sigma_0=\sigma_1=1$, $h_0(t)=h_1(t)=t$, $b_0(t)=0$, $b_1(t)=3$, i.e.,

$$\frac{\partial T}{\partial n}(0,t)+T(0,t)=f_0 t, \quad \frac{\partial T}{\partial n}(1,t)+T(1,t)=f_1 t+3, \quad t\in(0,1], \tag{5.2}$$

and the additional measurements (3.3) taken at $t^0=\tilde{t}_{i_0}$ with $i_0\in\{1,..,N\}$ fixed, namely

$$T(0,\tilde{t}_{i_0})=2\tilde{t}_{i_0}=\chi_0, \quad T(1,\tilde{t}_{i_0})=1+2\tilde{t}_{i_0}=\chi_1. \tag{5.3}$$

It can easily be seen that the conditions of Theorem 3.1 are satisfied and therefore, a solution to the IHCP given by (2.1), (5.1–5.3) is unique. Further, this analytical solution $(T(x,t),\ f_0,\ f_1)$ to be determined is given by $T(x,t)=x^2+2t$, $f_0=f_1=2$. Noise is introduced in the measurement (5.3) by replacing χ_i with $\chi_i(1+\rho)$ for $i=0,1$, where ρ is the percentage of noise. The condition number of the matrix $\tilde{X}$, Cond$(\tilde{X})$, in (4.14) has been calculated

Table 2 The condition number of the matrix $\tilde{X}$, Cond($\tilde{X}$), and the constants f_0 and f_1, when $i_0 = 1$ for various (N_0, N) (no noise)

N_0		$N = 20$	$N = 40$	$N = 80$	$N = 160$
	Cond($\tilde{X}$)	8×10^4	8×10^5	9×10^6	1×10^8
	f_0	1.9527	1.8671	1.6252	0.9415
20	f_1	2.2876	3.2053	6.9189	22.4320
	f_0	1.9876	1.9662	1.9054	1.7342
40	f_1	2.0723	2.3023	3.2452	7.0871
	f_0	1.9964	1.9909	1.9755	1.9323
80	f_1	2.0186	2.0770	2.3142	3.2769
	f_0	1.9986	1.9971	1.9930	1.9819
160	f_1	2.0051	2.0206	2.0817	2.3259

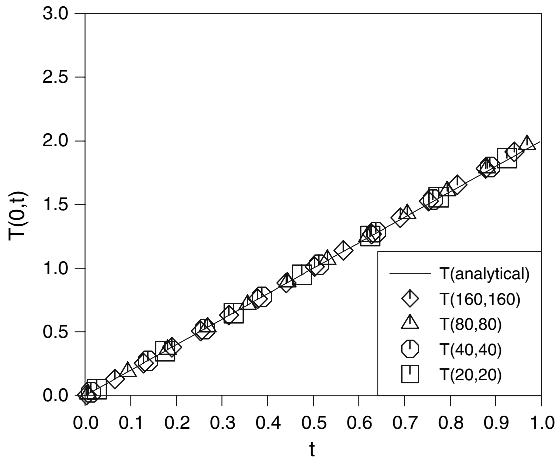

Fig. 1 The numerical and analytical boundary temperatures $T(0, t)$, as functions of time t, when $i_0 = 1$ for various (N_0, N) (no noise)

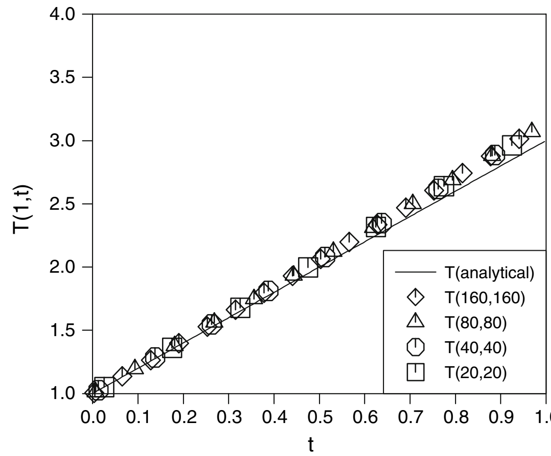

Fig. 2 The numerical and analytical boundary temperatures $T(1, t)$, as functions of time t, when $i_0 = 1$ for various (N_0, N) (no noise)

using the NAG routine `F07AGF`. The BEM mesh has been taken uniform, i.e., $x_k = k/N_0$, $k = \overline{0, N_0}$, $t_j = jt_f/N$, $j = \overline{0, N}$.

In Table 2, we find that increasing the number of time elements N from 20 to 160, when $i_0 = 1$, results in condition numbers of matrix $\tilde{X}$ increasing significantly. The error in f_0 is far less than that in f_1, a fact that is attributed to the smaller value of the additional measurement χ_0 in comparison to χ_1. However, for a fixed number of time elements N, the accuracy in the numerical approximations improve when increasing the number of space cells N_0. This is because: (i) the number of space cells increases the accuracy of the approximation (4.10), and (ii) the boundary values χ_0 and χ_1 are measured at the same point from the initial condition, thus the ill-conditioning of the system of equations remains unchanged. The best results are obtained when $N = 20$ and $N_0 = 160$.

Figures 1–4 show the boundary temperature and the heat flux for various values of (N_0, N). Reasonable numerical approximations to the exact solutions are obtained.

Figure 5 shows the temperature contours when the measurement (3.3) is taken at $i_0 = 1$ in the absence of noise. We observe some inaccuracies in the temperature, especially with increasing time, and particularly towards the boundary $x = 1$ where the maximum of the $T(x, t) = x^2 + 2t$ occur.

In Table 3, when $i_0 = 1$ and noise is introduced in the data (3.3), as expected, the approximated values of the constants f_0 and f_1 blow up, but they improve slightly with the increasing value of N_0, when N is fixed. On the other hand, the inaccuracy in the numerical approximations of f_0 and f_1 worsen with increasing N, for a fixed N_0. These inaccurate results which are full of significant jumps in the values of f_0 and f_1 demonstrate the instability of

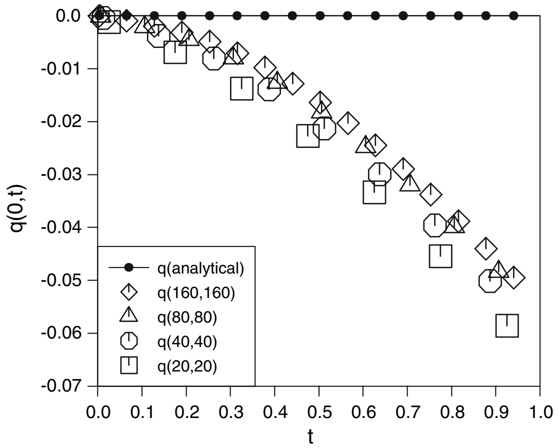

Fig. 3 The numerical and analytical heat fluxes $q(0,t) = \frac{\partial T}{\partial n}(0,t)$, as functions of time t, when $i_0 = 1$ for various (N_0, N) (no noise)

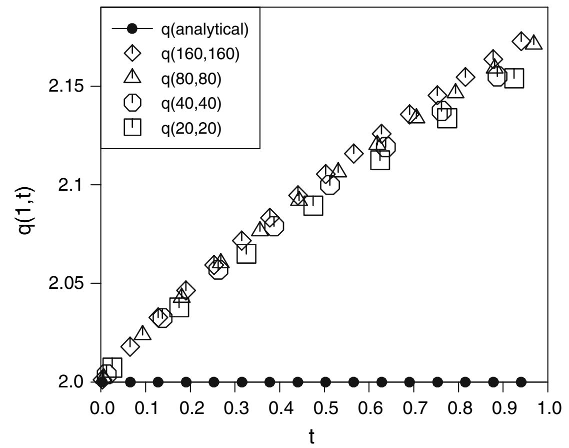

Fig. 4 The numerical and analytical heat fluxes $q(1,t) = \frac{\partial T}{\partial n}(1,t)$, as functions of time t, when $i_0 = 1$ for various (N_0, N) (no noise)

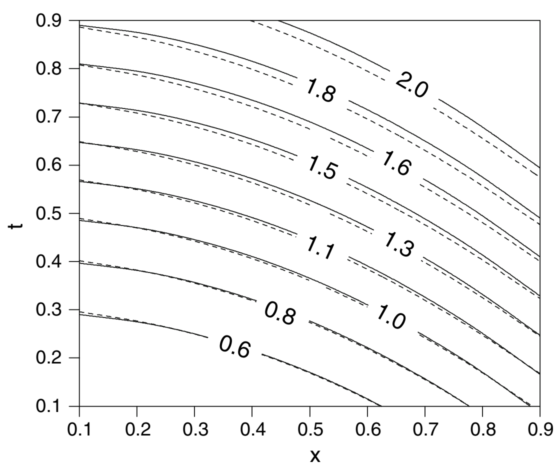

Fig. 5 The numerical (. . .) and the analytical (—) temperature contours in the domain $(x,t) \in (0.1, 0.9) \times (0.1, 0.9)$, when $i_0 = 1$ and $(N_0, N) = (20, 20)$ (no noise)

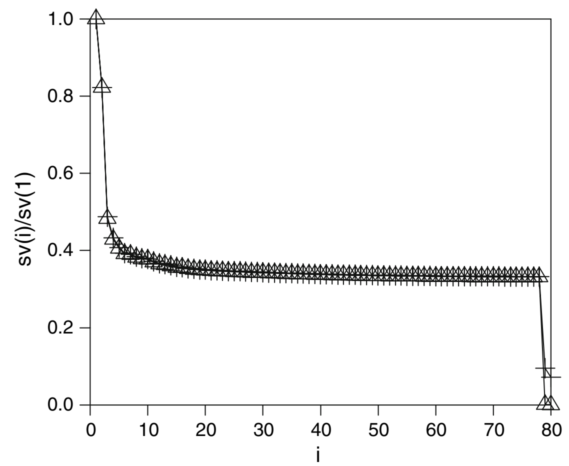

Fig. 6 The normalised singular values $\frac{\mathrm{sv}(i)}{\mathrm{sv}(1)}$, for the IHCPs given by Example 1, when $(N_0, N) = (40, 40)$, as a function of i, when $i_0 = 1(\triangle)$ and $i_0 = N(+)$

the numerical solution of the system of equations (4.14) for the IHCP given by Example 1 when $i_0 = 1$, since the measurement time $\tilde{t}_{i_0} = \tilde{t}_1 = 1/(2N)$ is too close to the initial time $t = 0$ for sufficient additional propagation of information to have been recorded yet. Finally, we note that using Tikhonov's regularization method of zeroth and first-order taking $10^{-12} < \kappa < 10^{-1}$ as regularisation parameter did not improve the results in predicting f_0 and f_1. Similarly, when the last two smallest singular values are zeroed, the use of truncated SVD did not also improve the numerical results. In Fig. 6, the last two singular values are near zero when $i_0 = 1$ and thus they increase the condition number significantly, whereas when $i_0 = N$, the matrix $\tilde{X}$ has normalised singular values reducing from 1 to approximately 0.1, which is not very low hence, the system becomes well-conditioned.

In Table 4, when $i_0 = N$, the condition numbers of the system remain low, and the approximate values of f_0 and f_1 are stable and accurate being the same for all values of N_0 and being almost the same for all values of N, when 1% noise is introduced in (3.3).

Table 3 The constants f_0 and f_1, when $i_0 = 1$ for various (N_0, N) (1% noise)

N_0		$N = 20$	$N = 40$	$N = 80$	$N = 160$
	f_0	2.0847	2.0456	1.8694	1.2786
20	f_1	5.0617	10.5252	26.7621	76.6997
	f_0	2.1198	2.1447	2.1497	2.0713
40	f_1	4.846	9.6222	23.0254	61.3548
	f_0	2.1285	2.1694	2.2197	2.2694
80	f_1	4.7927	9.3968	22.0944	57.5445
	f_0	2.1307	2.1756	2.2372	2.3189
160	f_1	4.7793	9.3405	21.8618	56.5935

Table 4 The condition number of the matrix $\tilde{X}$, Cond($\tilde{X}$), and the constants f_0 and f_1, when $i_0 = N$ and $N_0 \in \{20, 40, 80, 160\}$ for various N (1% noise)

	$N = 20$	$N = 40$	$N = 80$	$N = 160$
Cond($\tilde{X}$)	705.4	2576.7	9910.3	3911.7
f_0	2.0268	2.0266	2.0666	2.0265
f_1	2.0593	2.0588	2.0585	2.0584

Next, we maintain the number of time steps and space cells fixed at $(N, N_0) = (40, 40)$, but we vary the value of i_0 at which the measurements (3.3) are taken.

Numerical approximations obtained when 1% noisy temperature measurements (3.3) are taken at various values of $i_0 \in \{1, \ldots, N\}$ are shown in Table 5. In Table 5 the condition numbers when $i_0 = 1$ and $i_0 = N = 40$ are very large and low, respectively. This shows that the system of equations (4.14) is well-conditioned when $i_0 = N = 40$ and gradually becomes ill-conditioned when reducing i_0 to 1, such that the approximate values of the constants f_0 and f_1, when i_0 approaches N become more accurate, but become significantly inaccurate when i_0 approaches 1; see Table 5. The condition number drops sharply from approximately 89.176×10^4 for $i_0 = 1$ to 0.258×10^4 for $i_0 = N$. However, the numerical approximations of the solutions when $i_0 = 1$ were significantly inaccurate due to the failure of the unstable inversion $\underline{\tilde{Y}}=\tilde{X}^{-1}\underline{\tilde{Z}}$ of the Gaussian elimination method, such that during computation more noise filters back into the system, causing an amplification effect of the error. In order to investigate in more detail the case i_0, we have re-run the BEM computational program for $t_f = 1/79$, and $N = N_0 = 40$, such that the old $\tilde{t}_{i_0}$ for $t_f = 1$, becomes now a new $\tilde{t}_{i_N}$ for $t_f = 1/79$. The resulting approximations of the constants f_0 and f_1 when noise and no noise is introduced into the additional condition are not improved when compared with the corresponding approximations in Tables 2 and 3.

Finally, Table 6 presents the condition number of the matrix $\tilde{X}$ and the constants f_0 and f_1, when, instead of (3.3), the additional measurements (3.4) with $e_0 = 1$ and $e_1 = 2$ are imposed, both with 1% noise and without noise. This results into a relatively stable system of equations (4.14), with low condition number and generation of accurate and stable approximations of the constants f_0 and f_1. Other test examples have been investigated producing the same qualitative conclusions.

6 Problem II

In Problem II, the function $f(x, t)$, $x \in \{0, 1\}$, $t \in (0, t_f]$ depends on t only, i.e., $f(0, t) = f(1, t) =: f(t)$. Denoting $\sigma_0(t) := \sigma(0, t)$ and $\sigma_1(t) := \sigma(1, t)$ the boundary conditions (2.3) and (2.4) become

$$\frac{\partial T}{\partial n}(0, t) + \sigma_0(t)T(0, t) = h_0(t)f(t) + b_0(t), \quad \text{for} \quad t \in (0, t_f], \tag{6.1}$$

$$\frac{\partial T}{\partial n}(1, t) + \sigma_1(t)T(1, t) = h_1(t)f(t) + b_1(t), \quad \text{for} \quad t \in (0, t_f], \tag{6.2}$$

Table 5 The variation of the condition number of the matrix $\tilde{X}$, Cond$(\tilde{X})$, and the constants f_0 and f_1, as a function of $i_0 = \overline{1, N}$, when $(N_0, N) = (40, 40)$ (1% noise)

i_0	Cond$(\tilde{X})$ $(\times 10^4)$	f_0	f_1
1	89.176	2.1447	9.6222
2	24.064	2.1371	4.1400
4	7.472	2.0990	2.7171
8	2.530	2.0672	2.2874
12	1.364	2.0517	2.1804
16	0.886	2.0430	2.1332
20	0.640	2.0375	2.1069
24	0.495	2.0338	2.0902
28	0.402	2.0312	2.0786
32	0.338	2.0292	2.0702
36	0.293	2.0278	2.0638
39	0.270	2.0269	2.0599
40	0.258	2.0266	2.0588

Table 6 The condition number of the matrix $\tilde{X}$, Cond$\tilde{X}$, and the constants f_0 and f_1 when the additional measurements (3.4) instead of (3.3) are imposed, for various (N_0, N) (no noise and 1% noise)

N_0, N	Cond$(\tilde{X})$ $(\times 10^4)$		$\rho = 0.00$	$\rho = 0.01$
20,20	13.30	f_0	1.9998	2.0288
		f_1	2.0002	2.0964
40,40	26.98	f_0	1.9998	2.0292
		f_1	2.0000	2.0963
80,80	54.83	f_0	1.9999	2.0293
		f_1	2.0000	2.0963
160,160	111.30	f_0	1.9999	2.0293
		f_1	1.9999	2.0963

where $\sigma_i(t)$, $h_i(t)$, $b_i(t)$ are given functions of time, $i = 0, 1$, but the function $f(t)$ is unknown. The additional information is given by the boundary temperature measurement

$$T(x, t) = \overline{\chi}(t), \quad \text{for} \quad t \in [0, t_f], \tag{6.3}$$

where $x = 0$ or $x = 1$.

Alternatively, instead of (6.3) we can measure the boundary observation

$$\gamma_0 T(0, t) + \gamma_1 T(1, t) = \overline{\chi}(t), \quad \text{for } t \in [0, t_f], \tag{6.4}$$

where γ_0 and γ_1 are given constants. Conditions (6.3) and (6.4) are called a point and an integral boundary observation, respectively.

We denote the solution of the direct problem (2.1), (2.2), and (6.1) and (6.2) with $f = 0$, i.e.,

$$\frac{\partial T^0}{\partial n}(0, t) + \sigma_0(t) T^0(0, t) = b_0(t), \quad \text{for } t \in (0, t_f], \tag{6.5}$$

$$\frac{\partial T^0}{\partial n}(1, t) + \sigma_1(t) T^0(1, t) = b_1(t), \quad \text{for } t \in (0, t_f], \tag{6.6}$$

by $T^0(x, t)$, and introduce the function $\chi(t) = \overline{\chi}(t) - T^0(x, t)$, where $x = 0$ or $x = 1$ for condition (6.3), and $\chi(t) = \overline{\chi}(t) - \gamma_0 T^0(0, t) - \gamma_1 T^0(1, t)$ for condition (6.4). We also introduce the condition

$$\chi \in C^{1/2}([0, t_f]), \quad F(t) := \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{\chi(\tau)}{\sqrt{t - \tau}} \mathrm{d}\tau \in C([0, t_f]), \tag{6.7}$$

where C^α is the space of Hölder continuous functions with exponent α. Then we have the following existence, uniqueness and stability theorem.

Theorem 6.1 ([17]) *Suppose $g \in C^1([0,1])$, σ_i, h_i, $b_i \in C([0,t_f])$, $i = 0, 1$ and $h_0(t) \neq 0$ or $h_1(t) \neq 0$ for condition (6.3), or $\gamma_0 h_0(t) + \gamma_1 h_1(t) \neq 0$ for condition (6.4), for all $t \in [0, t_f]$. Further, suppose that condition (6.7) is satisfied. Then there exists a unique solution $(T(x,t) \in C^{2,1}(Q),\ f \in C([0,t_f]))$ of the inverse problem (2.1), (2.2), (6.1–6.3), or (2.1), (2.2), (6.1), (6.2), (6.4). Furthermore, the stability conditions*

$$||f|| + ||T - T^0|| \leq C||F||, \tag{6.8}$$

$$||f|| + ||T|| \leq C\left(||g|| + ||b_0|| + ||b_1|| + ||\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{\overline{\chi}(\tau)}{\sqrt{t-\tau}}\mathrm{d}\tau||\right) \tag{6.9}$$

for some positive constant C, are valid, where the norms are in the space of continuous functions.

From Theorem 6.1 it follows that under its hypotheses the inverse Problem II is solvable and well-posed, i.e., it is also stable in the appropriate topology, as given by the stability estimates (6.8) and (6.9).

7 The BEM for Problem II

Now instead of (4.9) we have the discretised version of (6.1) and (6.2), namely,

$$T'_{0i} + \sigma_{0i}T_{0i} = h_{0i}f_i + b_{0i},\ \ T'_{1i} + \sigma_{1i}T_{1i} = h_{1i}f_i + b_{1i}, \quad i = \overline{1, N}, \tag{7.1}$$

where $\sigma_{0i} = \sigma_0(\tilde{t}_i)$, $\sigma_{1i} = \sigma_1(\tilde{t}_i)$ and $f_i = f(\tilde{t}_i)$. Also, the discretised versions of (6.3) and (6.4) read as

$$T_{0i} = \overline{\chi}_i \quad \text{or} \qquad T_{1i} = \overline{\chi}_i, \quad i = \overline{1, N}, \tag{7.2}$$

and

$$\gamma_0 T_{0i} + \gamma_1 T_{1i} = \overline{\chi}_i, \qquad i = \overline{1, N}, \tag{7.3}$$

respectively, where $\overline{\chi}_i = \overline{\chi}(\tilde{t}_i)$ The IHCP given by (2.1), (2.2), (6.1), (6.2), and (6.3) or (6.4) reduces to its discretised version given by (4.8), (4.10), (7.1), and (7.2) or (7.3). Remark that from (7.1) we can eliminate f_i, i.e.,

$$f_i = \frac{T'_{0i} + \sigma_{0i}T_{0i} - b_{0i}}{h_{0i}} \quad \text{or} \quad \frac{T'_{1i} + \sigma_{1i}T_{1i} - b_{1i}}{h_{1i}} = f_i, \quad i = \overline{1, N}, \tag{7.4}$$

depending on which data $h_0(t)$ or $h_1(t)$ is non-zero on the interval $[0, t_f]$. Based on the above elimination process, the whole inverse problem can be reduced to a $3N \times 3N$ system of equations of the type (4.14) which in a generic form can be writen as

$$\tilde{\tilde{X}}\underline{\tilde{\tilde{Y}}} = \underline{\tilde{\tilde{Z}}}, \tag{7.5}$$

where the unknown vector $\underline{\tilde{\tilde{Y}}}$ contains the components of $(T'_{0i})_{i=\overline{1,N}}$, $(T'_{1i})_{i=\overline{1,N}}$ and $(T_{0i})_{i=\overline{1,N}}$ or $(T_{1i})_{i=\overline{1,N}}$, regarding whether they are known or unknown with respect to condition (6.3). Once $\underline{Y}$ is found, $(f_i)_{i=\overline{1,N}}$ can be obtained from (7.4).

8 Numerical results and discussion for Problem II

8.1 Example 2

In this example, we solve the IHCP given by the heat equation (2.1) in the domain $Q = (0, 1) \times (0, t_f = 1]$, subject to the initial condition (5.1), the boundary conditions (6.1) and (6.2) with $\sigma_0 = \sigma_1 = 1$, $h_0 = h_1 = 2$, $b_0 = 0$ and $b_1 = 3$, i.e.,

$$\frac{\partial T}{\partial n}(0,t) + T(0,t) = 2f(t), \quad \frac{\partial T}{\partial n}(1,t) + T(1,t) = 2f(t) + 3, \quad t \in (0, 1], \tag{8.1}$$

and the additional measurement (6.3) at $x = 0$, i.e.,

$$T(0, t) = \overline{\chi}(t) = 2t, \quad t \in [0, 1]. \tag{8.2}$$

To check that the hypotheses of Theorem 6.1 are satified and thus conclude the unique solvability of the inverse problem given by (2.1), (5.1), (8.1) and (8.2), we need first to compute the solution $T^0(x, t)$ of the direct problem (2.1), (5.1), (6.5) and (6.6), i.e.,

$$\frac{\partial T^0}{\partial t}(x, t) = \frac{\partial^2 T^0}{\partial x^2}(x, t), \quad (x, t) \in (0, 1) \times (0, 1], \tag{8.3}$$

$$T^0(x, 0) = x^2, \quad x \in [0, 1], \tag{8.4}$$

$$\frac{\partial T^0}{\partial n}(0, t) + T^0(0, t) = 0, \quad \frac{\partial T^0}{\partial n}(1, t) + T^0(1, t) = 3, \quad t \in (0, 1]. \tag{8.5}$$

The analytical solution of the problem (8.3–8.5) is given by, [26, pp. 114–118],

$$T^0(x, t) = x + 1 + 2 \sum_{n=1}^{\infty} \frac{\mathrm{e}^{-\alpha_n^2 t} \left(\alpha_n \cos(\alpha_n x) + \sin(\alpha_n x)\right)}{\alpha_n^2 + 3} \int_0^1 (x^2 - x - 1) \left(\alpha_n \cos(\alpha_n x) + \sin(\alpha_n x)\right) \mathrm{d}x, \tag{8.6}$$

where α_n are the positive real roots of the transcedental equation

$$\tan(\alpha) = \frac{2\alpha}{\alpha^2 - 1}. \tag{8.7}$$

Remarking that from (8.7) we have

$$\sin(\alpha) = \frac{2\alpha}{\alpha^2 + 1}, \qquad \cos(\alpha) = \frac{\alpha^2 - 1}{\alpha^2 + 1}, \tag{8.8}$$

and performing the integration in (8.6), we obtain

$$T^0(x, t) = x + 1 - 8 \sum_{n=1}^{\infty} \frac{1}{\alpha_n^3(\alpha_n^2 + 3)} \left(\alpha_n \cos(\alpha_n x) + \sin(\alpha_n x)\right) \mathrm{e}^{-\alpha_n^2 t}. \tag{8.9}$$

From (8.4) and (8.9) we obtain the important identity

$$\frac{x + 1 - x^2}{8} = \sum_{n=1}^{\infty} \frac{\alpha_n \cos(\alpha_n x) + \sin(\alpha_n x)}{\alpha_n^3(\alpha_n^2 + 3)}, \quad x \in [0, 1]. \tag{8.10}$$

Differentiating twice with respect to x and setting $x = 0$, we obtain the identity $\frac{1}{4} = \sum_{n=1}^{\infty} \frac{1}{\alpha_n^2 + 3}$. Using (8.2) and (8.9) we have

$$\chi(t) = \overline{\chi}(t) - T^0(0, t) = 2t - 1 + 8 \sum_{n=1}^{\infty} \frac{1}{\alpha_n^2(\alpha_n^2 + 3)} \mathrm{e}^{-\alpha_n^2 t}. \tag{8.11}$$

Clearly, $\chi \in C^{\frac{1}{2}}([0, 1])$ and $\chi(0) = 0$ since from (8.4) we have $T^0(0, 0) = 0$ and from (8.2) we have $\overline{\chi}(0) = 0$. Consider now the function F defined in (6.7), namely,

$$\begin{aligned}
F(t) &= \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{1}{\sqrt{t - \tau}} \left(2\tau - 1 + 8 \sum_{n=1}^{\infty} \frac{1}{\alpha_n^2(\alpha_n^2 + 3)} \mathrm{e}^{-\alpha_n^2 \tau}\right) \mathrm{d}\tau \\
&= 4\sqrt{t} - \frac{1}{\sqrt{t}} + 8 \sum_{n=1}^{\infty} \frac{1}{\alpha_n^2(\alpha_n^2 + 3)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{\mathrm{e}^{-\alpha_n^2 \tau}}{\sqrt{t - \tau}} \mathrm{d}\tau \\
&= 4\sqrt{t} - \frac{1}{\sqrt{t}} + \frac{8}{\sqrt{t}} \sum_{n=1}^{\infty} \frac{1}{\alpha_n^2(\alpha_n^2 + 3)} - 8\sqrt{\pi} \sum_{n=1}^{\infty} \frac{\mathrm{erfi}(\sqrt{t}\alpha_n)}{\alpha_n(\alpha_n^2 + 3)} \mathrm{e}^{-\alpha_n^2 t},
\end{aligned} \tag{8.12}$$

where erfi is the imaginary error function which for real value of x is defined as

$$\mathrm{erfi}(x)=\frac{2}{\sqrt{\pi}}\mathrm{e}^{x^2}\int_0^\infty \mathrm{e}^{-t^2}\sin(2xt)\mathrm{d}t.$$

Since $\chi(0)=0$, from (8.10), we have that $-1+8\sum_{n=1}^{\infty}1/(\alpha_n^2(\alpha_n^2+3))=0$, so that $F(t)=4\sqrt{t}-8\sqrt{\pi}\sum_{n=1}^{\infty}\mathrm{e}^{-\alpha_n^2 t}\mathrm{erfi}(\sqrt{t}\alpha_n)/(\alpha_n(\alpha_n^2+3))\in C([0,1])$.
The series in (8.12) is uniformly convergent on [0, 1] since

$$8\sqrt{\pi}\sum_{n=1}^{\infty}\frac{\mathrm{erfi}(\sqrt{t}\alpha_n)}{\alpha_n(\alpha_n^2+3)}\mathrm{e}^{-\alpha_n^2 t}=16\sum_{n=1}^{\infty}\frac{1}{\alpha_n(\alpha_n^2+3)}\int_0^{\alpha_n\sqrt{t}}\mathrm{e}^{\sigma^2-\alpha_n^2 t}\mathrm{d}\sigma$$
$$\leq 16\sum_{n=1}^{\infty}\frac{1}{\alpha_n(\alpha_n^2+3)}\int_0^{\alpha_n\sqrt{t}}\mathrm{d}\sigma=16\sqrt{t}\sum_{n=1}^{\infty}\frac{1}{\alpha_n^2+3}=4\sqrt{t}.$$

Therefore, the conditions of Theorem 6.1 are satisfied and hence the inverse problem (2.1), (5.1), (8.1) and (8.2), is solvable. It is easy to verify that the unique solution is given by $f(t)=t$, $T(x,t)=x^2+2t$.

Figure 7 shows the normalised singular values of the system of equations (7.5) of $3N$ equations with $3N$ unknowns $T_{1,i}$, $T'_{0,i}$ and $T'_{1,i}$, $i=\overline{1,N}$, when $(N_0,N)=(40,40)$. This BEM mesh was used in all the numerical results presented in the figures of this section. From Figure 7, it can be seen that the normalised singular values reduce from 1 to approximately 0.02, which gives the condition number equal to approximately $50=\mathrm{sv}(1)/\mathrm{sv}(120)$. Thus, the system of equations (7.5) is quite well-conditioned, as expected from the stability of the solution given in Theorem 6.1.

In order to test the stability of the numerical inversion, both additive and multiplicative noise are introduced in the measurement data (6.3). The additive noise is introduced as

$$T_{0i}^{\epsilon}=2\tilde{t}_i+\epsilon_i,\quad i=\overline{1,N}, \tag{8.13}$$

where ϵ_i are Gaussian random variables with zero mean and standard deviation 2ρ, where ρ is the percentage of noise, generated using the NAG routine `G05DDF`.

The multiplicative noise is introduced as

$$T_{0i}^{\epsilon}=2\tilde{t}_i(1+\rho\epsilon_i),\quad i=\overline{1,N}, \tag{8.14}$$

where ϵ_i are random variables taken from a uniform distribution in [−1,1], generated using the NAG routine `G05DAF`.

Figure 8 shows the analytical and the additive noisy $T(0,t)$, as functions of time t. We observe that the noisy $T(0,t)$ is more pronounced around $t_f=1$ and evenly distributed on either sides of the analytical curve in the other portions of the graph.

Figures 9–12 show the numerical solutions for $T(1,t)$, $q(0,t):=-\partial T/\partial x(0,t)$, $q(1,t):=\partial T/\partial x(1,t)$ and $f(t)$, respectively, when $\rho=0$, i.e., no noise, and $\rho=5\%$ additive noise is introduced into the data (8.13). For no noise, the numerical results are in excellent agreement with the exact solutions $T(1,t)=1+2t$, $q(0,t)=0$, $q(1,t)=2$ and $f(t)=t$. However, when noise is introduced, the numerical solutions shown by $(-\triangle-)$ have a more prononounced disagreement with the corresponding exact solutions, especially near $t=1$, because they are obtained from the input values $T(0,t)$ which also had larger errors near $t=1$, as shown in Fig. 8. Furthermore, as is expected, the heat-flux prediction is less accurate than the boundary-temperature prediction.

Although not illustrated, it is reported that some regularised features of the heat flux can be further obtained if one uses the truncated singular-value decomposition, for solving the direct ill-posed problem of retrieving higher-order (Neumann) derivatives from noisy lower-order (Dirichlet) data $T(0,t)$ and $T(1,t)$ shown by $(-\triangle-)$ in Figs. 8 and 9, respectively [27].

Finally, Table 7 compares the numerical solutions for the interior temperature when no noise, 1% and 5% additive noise is introduced into the temperature measurement as in (8.13), for various numbers of space cells and time boundary elements (N_0,N), in comparison with the analytical solution $T(0.5,0.5)=1.25$. The numerical

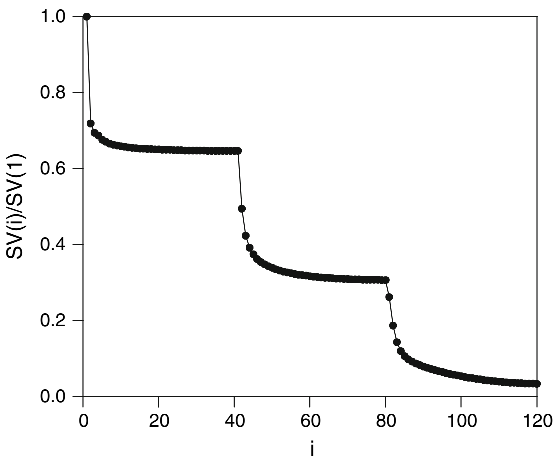

Fig. 7 The normalised singular values $\frac{\text{SV}(i)}{\text{SV}(1)}$ of the system of equations (7.5), as a function of $i = \overline{1, 3N}$

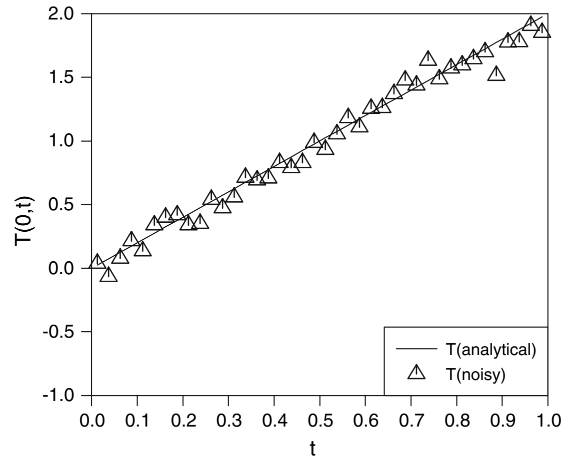

Fig. 8 The analytical and the 5% additive noisy boundary temperatures $T(0, t)$, as functions of time t

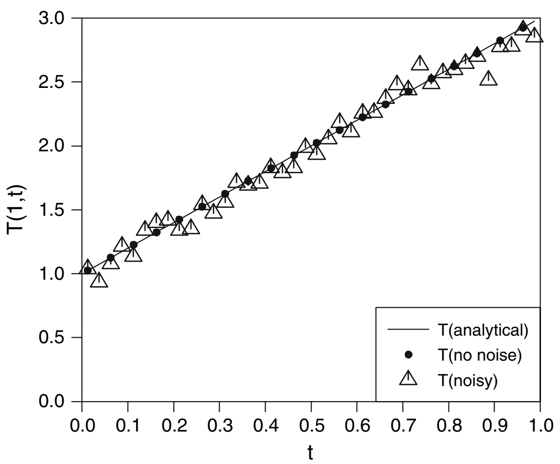

Fig. 9 The analytical and the numerical solutions for the boundary temperature $T(1, t)$, as functions of time t (5% additive noise)

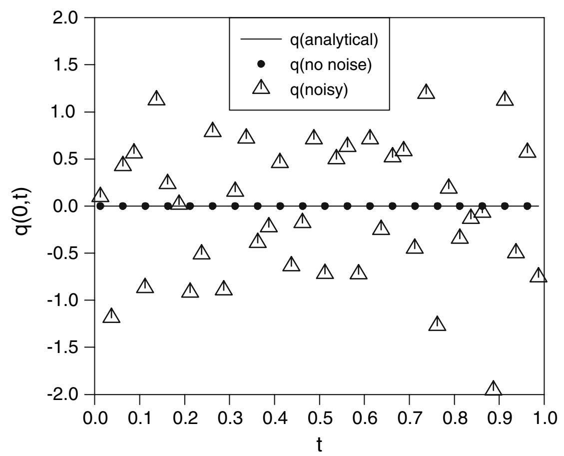

Fig. 10 The analytical and the numerical solutions for the heat flux $q(0, t)$, as functions of time t (5% additive noise)

results show very good agreement with the analytical solution for errorless data and the stability of the numerical solution for noisy data. The same conclusions are obtained when input data contaminated with the multiplicative noise (8.14), instead of the additive noise (8.13), are inverted.

9 Conclusions

Inverse problems in heat conduction which require finding the spacewise or time-dependent, ambient temperature appearing in the boundary conditions from additional terminal, integral or point observations have been investigated. Under these additional measurements (observations) solvability results are available [16,17]. The solutions of the inverse problems have been found numerically using the BEM. It was illustrated that the numerical BEM produced convergent and stable numerical results. In the spacewise-dependent ambient temperature case the ill-conditioning

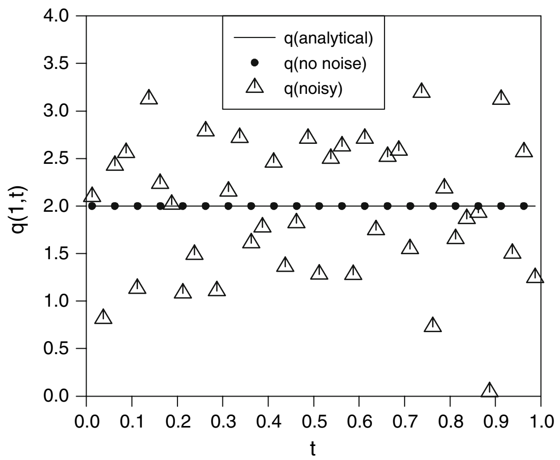

Fig. 11 The analytical and the numerical solutions for the heat flux $q(1, t)$, as functions of time t (5% additive noise)

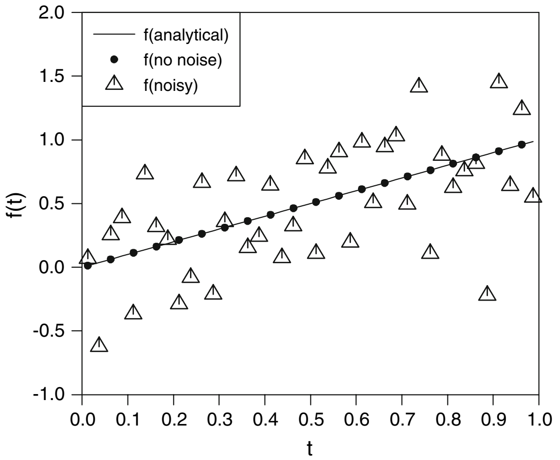

Fig. 12 The analytical and the numerical solutions for the function $f(t)$, as functions of time t (5% additive noise)

Table 7 The analytical and the numerical solutions for the interior temperature $T(0.5, 0.5)$, when additive noise and no noise is introduced in the measurement (8.13), for various (N_0, N)

N_0, N	T no noise	T 1% noise	T 5% noise	T analytical
20,20	1.25046	1.24302	1.21326	1.25000
40,40	1.25003	1.24030	1.20135	1.25000
80,80	1.25000	1.24241	1.21207	1.25000
160,160	1.25000	1.24693	1.23466	1.25000

of the system of linear equations decreases with increasing the instant at which the additional boundary-temperature measurements are made. Analogous inverse problems which require finding the spacewise or time-dependent heat-transfer coefficient are deferred to a future work. Future work will also involve extensions to higher dimensions in which the spacewise variation of the unknown coefficients in the boundary conditions becomes more meaningful than in the one-dimensional case in which two constants only had to be retrieved.

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